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Electromagnetic properties of a neutrino stream

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In a medium that contains a neutrino background in addition to the matter particles, the neutrinos contribute to the photon self-energy as a result of the effective electromagnetic vertex that they acquire in the presence of matter. We calculate the contribution to the photon self-energy in a dense plasma, due to the presence of a gas of charged particles, or neutrinos, that moves as a whole relative to the plasma. General formulas for the transverse and longitudinal components of the photon polarization tensor are obtained in terms of the momentum distribution functions of the particles in the medium, and explicit results are given for various limiting cases of practical interest. The formulas are used to study the electromagnetic properties of a plasma that contains a beam of neutrinos. The transverse and longitudinal photon dispersion relations are studied in some detail. Our results do not support the idea that neutrino streaming instabilities can develop in such a system. We also indicate how the phenomenon of optical activity of the neutrino gas is modified due to the velocity of the neutrino background relative to the plasma. The general approach and results can be adapted to similar problems involving relativistic plasmas and high-temperature gauge theories in other environments.

I. INTRODUCTION AND CONCLUSIONS

From a modern point of view, the methods of finite temperature field theory (FTFT) [1] provide a natural setting to treat the problems related to the propagation of photons and neutrinos through a dense medium. This view has been largely stimulated by the work of Weldon [2–4], who emphasized the convenience of carrying out covariant, real-time calculations in this kind of problem. The work of Weldon demonstrated that the real-time formulation of FTFT is well suited to the study of systems involving gauge fields and/or chiral fermions at finite temperature, which can be extended in an efficient and transparent way to realistic situations involving, for example, photons and/or neutrinos [5] in a matter background.

The electromagnetic properties of neutrinos in a medium, besides their intrinsic interest, are relevant in many physical applications [6]. For example, the induced electromagnetic couplings of a neutrino propagating in a background of electrons and nucleons is responsible for the plasmon decay process $\gamma \to \nu\bar{\nu}$ in stars, and modify the MSW resonant condition in the presence of an external magnetic field [7–10]. A neutrino gas also exhibits the phenomemon of optical activity as a result of the chiral nature of the neutrino interactions [11,12].

The covariance in this type of calculation is implemented by introducing the velocity four-vector u^μ of the medium, in terms of which the thermal propagators are written in a manifestly covariant form. Therefore, covariance is maintained, but quantities such as the photon self-energy or the neutrino electromagnetic form factors depend not just on the kinematical momentum variables of the problem, but also on u^μ. In practice the vector u^μ is in the end set to $(1,\vec{0})$, which is equivalent to having carried out the calculation from the start with respect to a frame of reference in which the medium is at rest. This is usually the relevant situation. Therefore, while generally useful, the covariant nature of these methods has not been of particular importance in the applications mentioned.

However, there is a class of problem in which setting $u^\mu = (1,\vec{0})$ is not possible. These are problems that involve one stream of particles (which we can think of as a moving medium) flowing through a background medium, which we can take to be stationary. If we denote by u^μ the velocity four-vector of the stationary medium, and by u'^μ the corresponding one for the moving background, then we can set $u^\mu = (1,\vec{0})$, but we cannot take both to be $(1,\vec{0})$ simultaneously. Thus, for example, if we were to calculate the self-energy of the photon propagating through such media, it will depend on the momentum and velocity four-vector u^μ as usual, and in addition on u'^μ. This additional dependence can have consequences that are as important as the effects of the stationary background itself.

For example, it is well known that in a plasma in which a bunch of electrons move, as a whole, relative to a plasma at rest, besides the usual dispersion relation of the longitudinal photon mode, another branch appears whose dispersion relation depends on the velocity of the beam. Under some conditions, the sign of the imaginary part of this dispersion relation is such that the corresponding amplitude grows exponentially, which signifies an instability of the system against the excitation of those modes. This kind of system is familiar in plasma physics research, and examples of them are discussed in textbooks on the subject [13,14].

Recently [15], it has been suggested that a similar kind of streaming stability might be driven by a flow of neutrinos through a matter background [16,17]. Because the neutrino acquires an effective electromagnetic coupling as it traverses a medium [18,19,7,10], the propagation of a photon in a medium that contains a drifting neutrino background may be affected in a way similar to the case mentioned above. As argued in Ref. [15], such effects can appear under the conditions of realistic situations such as those in a supernova explosion, gamma ray bursts, or the Early Universe.

Similarly, other neutrino processes that have been studied previously, such as those mentioned above, may be modified in important ways if the neutrino gas is moving as a whole relative to the matter background.

Motivated by all these considerations, in this work we calculate the neutrino contribution to the photon self-energy in a medium in which the neutrino gas moves as a whole relative to a matter background which we take to be at rest. The calculation is based on the application of FTFT to this problem in the manner that has been suggested above. The implicit assumption is that, in the rest frame of the stream, the neutrino background is characterized by a momentum distribution function that is parametrized in the usual way. Although our focus is the case in which the neutrino background constitutes the stream, largely motivated by the potential applications that have been mentioned, the calculation and the formulas for the photon self-energy are presented in such a way that they can be adapted to other cases as well. Therefore, they complement the existing calculations of the photon self-energy in which all the particles form a common background with a unique velocity four-vector. The results for the photon self-energy can be equivalently interpreted in terms of the dielectric constant of the system, and in that way we show that the well known textbooks results for the stream stabilities are reproduced when the appropriate limits are taken. On the other hand, the results we obtain are valid for general conditions (whether they are relativistic and/or degenerate or the converse) of the gases that form the plasma at rest as well as the stream, hold for general values of the velocity of the stream, and are valid also for general values of the photon momentum and not necessarily for some particular limit. Therefore, they are useful also in the study of similar processes that may occur in other contexts, such as high-temperature QCD, heavy ion collisions or other similar environments in which the methods of FTFT are applicable.

In Section II, we give the general one-loop formulas for the generic contribution of a moving fermion background to the photon self-energy. The contribution from any given fermion can be written in terms of a few independent functions, which are expressed as integrals over the momentum distribution functions of the fermion. Explicit formulas are given for various limiting cases of physical interest, which also serve to show how some of the results derived in textbooks for simple cases are reproduced in the appropriate limit.

The case of the system that is composed of a matter background made of electrons and nucleons (and possibly their antiparticles), and a neutrino gas that propagates as a whole relative to the matter background, is considered in Section III. We begin by reviewing the one-loop formulas for the effective electromagnetic vertex of the neutrino in a matter background, in the way that will be used in the calculation of the photon self-energy. The neutrino background contribution to the photon self-energy is then determined. It depends on the momentum distribution functions of the matter particles and well as those for the neutrinos. As an application of the formulas obtained, the dispersion relation for the longitudinal modes is considered, with attention to the possible effect of the neutrino contribution to the instability of the system. In that context, our results do not indicate the existence of unstable modes, and therefore we do not find support for the idea that stream instabilities due to the presence of the neutrino background can develop in such systems.

In Section III we also consider the dispersion relations for the transverse modes. The chiral nature of the neutrino interactions gives rise to the phenomenon of optical activity, which had been studied earlier [11,12]. Here we show how the results of Refs. [11,12] are modified when the neutrino gas is moving relative to the matter background. The main effect is that the dispersion relations for the two circularly polarized modes are not isotropic. As a consequence, the frequency splitting between them, which is the measure of the rotation of the plane of polarization, depends on the direction of propagation of the mode relative to the velocity of the neutrino gas. In particular, under the appropriate conditions, the frequency difference is the opposite to what is found if the neutrino gas is not moving relative to the matter background.

Section IV contains our outlook, where other possible effects and applications are also mentioned.

II. PHOTON SELF-ENERGY IN A FERMION BACKGROUND

We will consider a medium that consists of a gas of nucleons, electrons, neutrinos and their antiparticles. Each fermion gas gives a contribution to the elements of the 2×2 photon self-energy matrix, that are determined by calculating the diagram shown in Fig. 1.

$$i\pi^{(ab)}_{\mu\nu}(q) =$$

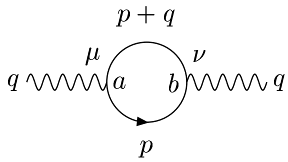

FIG. 1. Diagram for the contribution to the photon self-energy matrix from a generic fermion. For a charged fermion, the electromagnetic coupling is given by the tree-level terms in the Lagrangian while for the neutrino it is the one-loop vertex function induced by the matter background.

In particular, the contribution to the $\pi_{11\mu\nu}$ element from each fermion f in the loop is

$$i\pi^{(f)}_{11\mu\nu} = (-1)(-i)^2 \mathrm{Tr} \int \frac{d^4p}{(2\pi)^4} j^{(em)}_{f\mu}(q) iS^{(f)}_{F11}(p+q) j^{(em)}_{f\nu}(-q) iS^{(f)}_{F11}(p) \,, \tag{2.1}$$

where $j^{(em)}_{f\mu}(q)$ is the electromagnetic current of the fermion. It is defined in such a way that the on-shell matrix element of the electromagnetic current operator $J^{(em)}_{\mu}$ is given by

$$\langle f(p')|J^{(em)}(0)_{\mu}|f(p)\rangle = \bar{u}(p') j^{(em)}_{f\mu}(q) u(p) \,, \tag{2.2}$$

where $q = p - p'$ and $u(p)$ is a Dirac spinor. For the electron it is simply $e\gamma_\mu$, for the nucleons it must in principle include the magnetic moment term, and for the neutrino we must use the effective electromagnetic neutrino vertex in the medium. The fermion propagator that appears in Eq. (2.1) is given by

$$S^{(f)}_{F11}(p) = (\not{p} + m_f) \left[\frac{1}{p^2 - m_f^2 + i\epsilon} + 2\pi i \delta(p^2 - m_f^2) \eta_f(p \cdot u^{(f)}) \right] \tag{2.3}$$

where

$$\eta_f(p \cdot u^{(f)}) = \frac{\theta(p \cdot u^{(f)})}{e^{\beta_f p \cdot u^{(f)} - \alpha_f} + 1} + \frac{\theta(-p \cdot u^{(f)})}{e^{-\beta_f p \cdot u^{(f)} + \alpha_f} + 1} \,, \tag{2.4}$$

with β_f being the inverse temperature and α_f the fermion chemical potential. The vector $u^{(f)\,\mu}$ is the velocity four-vector of the fermion gas as a whole, so that $u^{(f)\,\mu} = (1, \vec{0})$ if the fermion background is at rest. In Eq. (2.4) we are allowing for the possibility that the different fermion gases of the background may be at different temperatures and, most importantly for our purposes later, that each gas has a velocity four-vector that is not necessarily the same for all of them. The implicit assumption here is that, in the rest frame of each fermion background, the corresponding particles have an isotropic thermal distribution characterized by a temperature and chemical potential $1/\beta_f$ and α_f.

The dispersion relations of the propagating photon modes are obtained by solving the equation

$$(q^2 g_{\mu\nu} - q^\mu q^\nu - \pi^{(\mathit{eff})}_{\mu\nu}) A^\nu = 0 \,, \tag{2.5}$$

where

$$\mathrm{Re}\, \pi^{(\mathit{eff})}_{\mu\nu} = \sum_f \mathrm{Re}\, \pi^{(f)}_{\mu\nu} \,, \tag{2.6}$$

and we have denoted by $\pi^{(f)}_{\mu\nu}$ the background-dependent term of Eq. (2.1). In the rest of this paper we will focus only on the real part of the dispersion relations, but similar considerations could be used to calculate the imaginary part as well.

In order to calculate $\mathrm{Re}\, \pi^{(\mathit{eff})}_{\mu\nu}$, and thus determine the dispersion relations, we must know the composition of the medium and the formulas for the electromagnetic current that must be substituted in Eq. (2.1). To proceed, we consider the various cases separately.

A. Matter background

We consider first an isotropic medium composed of nucleon and electron gases, with a common velocity four-vector u^μ. The most general form of the physical self-energy function in this case, which we denote by $\pi^{(m)}_{\mu\nu}$ is [2,11]

$$\pi^{(m)}_{\mu\nu} = \pi^{(m)}_T R_{\mu\nu}(q,u) + \pi^{(m)}_L Q_{\mu\nu}(q,u) + \pi^{(m)}_P P_{\mu\nu}(q,u)\,, \tag{2.7}$$

where

$$\begin{aligned} R_{\mu\nu}(q,u) &= g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2} - Q_{\mu\nu}(q,u) \\ Q_{\mu\nu}(q,u) &= \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} \\ P_{\mu\nu}(q,u) &= \frac{i}{\mathcal{Q}} \epsilon_{\mu\nu\alpha\beta} q^\alpha u^\beta\,, \end{aligned} \tag{2.8}$$

with

$$\tilde{u}_\mu \equiv \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) u^\nu\,. \tag{2.9}$$

Although we have not indicated it explicitly, in general, $\pi^{(m)}_{T,L,P}$ are functions of the scalar variables

$$\begin{aligned} \Omega &= q \cdot u \\ Q &= \sqrt{\Omega^2 - q^2}\,, \end{aligned} \tag{2.10}$$

which have the interpretation of being the photon energy and momentum in the rest frame of the medium.

A detailed calculation of the photon self-energy in such a medium was carried out in Ref. [20]. For our present purposes, it is useful to summarize those results as follows. The nucleon magnetic moment term contribution is not important for practical purposes. Therefore, we use here $j^{(em)}_{f\mu} = e_f \gamma_\mu$, so that the neutron contribution is being neglected. Substituting in Eq. (2.1) the formula for $S^{(f)}_{F11}$, the contribution from each fermion in the loop can be expressed in the form

$$\mathrm{Re}\,\pi^{(f)}_{\mu\nu} = -4e^2 \left[\frac{1}{2} \left(A_f(\Omega,\mathcal{Q}) - \frac{B_f(\Omega,\mathcal{Q})}{\tilde{u}^2} \right) R_{\mu\nu}(q,u) + \frac{B_f(\Omega,\mathcal{Q})}{\tilde{u}^2} Q_{\mu\nu}(q,u) \right]\,, \tag{2.11}$$

with the coefficients A_f and B_f defined as

$$\begin{aligned} A_f(\Omega,\mathcal{Q}) &= \int \frac{d^3p}{(2\pi)^3 2E_f} \Big(f_f(p\cdot u) + f_{\overline{f}}(p\cdot u) \Big) \left[\frac{2m_f^2 - 2p\cdot q}{q^2 + 2p\cdot q} + (q \to -q) \right] \\ B_f(\Omega,\mathcal{Q}) &= \int \frac{d^3p}{(2\pi)^3 2E_f} \Big(f_f(p\cdot u) + f_{\overline{f}}(p\cdot u) \Big) \left[\frac{2(p\cdot u)^2 + 2(p\cdot u)(q\cdot u) - p\cdot q}{q^2 + 2p\cdot q} + (q \to -q) \right]\,. \end{aligned} \tag{2.12}$$

In these formulas,

$$p^\mu = (E, \vec{p})\,, \quad E_f = \sqrt{\vec{p}^2 + m_f^2}\,, \tag{2.13}$$

and $f_{f,\overline{f}}$ denote the particle and antiparticle number density distributions

$$f_{f,\overline{f}}(\mathcal{E}) = \frac{1}{e^{\beta_f \mathcal{E} \mp \alpha_f} + 1} \tag{2.14}$$

with the minus(plus) sign holding for the particles(antiparticles), respectively. Comparing Eqs. (2.7) and (2.11) we can identify the contribution of any fermion to the real part of the transverse and longitudinal components of the self-energy, and therefore obtain

$$\begin{aligned} \mathrm{Re}\,\pi^{(m)}_T &= -2e^2 \sum_f \left(A_f(\Omega,\mathcal{Q}) + \frac{q^2}{Q^2} B_f(\Omega,\mathcal{Q}) \right)\,, \\ \mathrm{Re}\,\pi^{(m)}_L &= 4e^2 \sum_f \frac{q^2}{Q^2} B_f(\Omega,\mathcal{Q})\,, \end{aligned} \tag{2.15}$$

where the relation $\tilde{u}^2 = -\mathcal{Q}^2/q^2$ has been used.

B. Matter background and a stream of charged particles

We now consider a medium that contains, in addition to the background as has been considered above, another gas of particles with a velocity four-vector u'_μ. We will refer to them as the matter background and the stream, respectively, and we assume that the latter consists of only one specie of fermions f' with an electromagnetic coupling $j^{(em)}_{f'\mu} = e_{f'}\gamma_\mu$. The fermion f' could be, for example, the electron or any other charged particle. We will denote by U'^0 and $\vec{U}'$ the components of u'^μ in the rest frame of the matter background so that, in that frame,

$$u^\mu = (1, \vec{0}) \,, \qquad u'^\mu = (U'^0, \vec{U}') \,. \tag{2.16}$$

The contribution from f' to the photon-self-energy is given by a formula analogous to Eq. (2.11),

$$\pi^{(f')}_{\mu\nu} = \pi^{(f')}_T R_{\mu\nu}(q, u') + \pi^{(f')}_L Q_{\mu\nu}(q, u') \,, \tag{2.17}$$

where

$$\begin{aligned}
\mathrm{Re}\,\pi^{(f')}_T &= -2e^2_{f'}\left(A_{f'}(\Omega', \mathcal{Q}') + \frac{q^2}{\mathcal{Q}'^2} B_{f'}(\Omega', \mathcal{Q}')\right) \\
\mathrm{Re}\,\pi^{(f')}_L &= 4e^2_{f'}\frac{q^2}{\mathcal{Q}'^2} B_{f'}(\Omega', \mathcal{Q}') \,.
\end{aligned} \tag{2.18}$$

In Eq. (2.18) the functions $A_{f'}$ and $B_{f'}$ are given by formulas analogous to Eq. (2.12), but with the replacement $u^\mu \to u'^\mu$, and we have used $\tilde{u}'^2 = -\mathcal{Q}'^2/q^2$ where, similarly to Eq. (2.9),

$$\tilde{u}'_\mu = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) u'^\nu \,. \tag{2.19}$$

In analogy with Eq. (2.10), the variables Ω', $\mathcal{Q}'$ are defined by

$$\begin{aligned}
\Omega' &= q \cdot u' \\
\mathcal{Q}' &= \sqrt{\Omega'^2 - q^2} \,,
\end{aligned} \tag{2.20}$$

and they are expressed in terms of Ω and $\mathcal{Q}$ by the relations

$$\begin{aligned}
\Omega' &= U'^0\Omega - \vec{U}' \cdot \vec{\mathcal{Q}} \\
\mathcal{Q}' &= \sqrt{\left(U'^0\Omega - \vec{U}' \cdot \vec{\mathcal{Q}}\right)^2 - \Omega^2 + \mathcal{Q}^2} \,.
\end{aligned} \tag{2.21}$$

The total photon self-energy is given by

$$\pi^{(eff)}_{\mu\nu} = \pi^{(m)}_T R_{\mu\nu}(q, u) + \pi^{(m)}_L Q_{\mu\nu}(q, u) + \pi^{(f')}_T R_{\mu\nu}(q, u') + \pi^{(f')}_L Q_{\mu\nu}(q, u') \,. \tag{2.22}$$

Eq. (2.22) can be written in a convenient form by the following procedure. From the definition of $R_{\mu\nu}$ given in Eq. (2.8) we have

$$R_{\mu\nu}(q, u') = R_{\mu\nu}(q, u) + Q_{\mu\nu}(q, u) - Q_{\mu\nu}(q, u') \,. \tag{2.23}$$

We now define the vectors

$$e^\mu_1 \equiv \frac{R^{\mu\nu}(q, u)u'_\nu}{\sqrt{N_1}} \,, \qquad e^\mu_2 \equiv -iP^{\mu\nu}(q, u)e_{1\nu} \,. \tag{2.24}$$

with

$$\begin{aligned}
N_1 &= -u'_\mu u'_\nu R^{\mu\nu}(q, u) \\
&= \frac{(\tilde{u} \cdot u')^2}{\tilde{u}^2} - \tilde{u}'^2 \,,
\end{aligned} \tag{2.25}$$

which can be expressed in the form

$$N_1 = U'^2 \mathcal{Q}^2 - \left(\vec{U}' \cdot \vec{\mathcal{Q}}\right)^2 . \tag{2.26}$$

It is easy to verify that $e^\mu_{1,2}$ are mutually orthogonal and satisfy

$$e_{1,2} \cdot q = e_{1,2} \cdot \tilde{u} = 0 , \qquad e^2_{1,2} = -1 . \tag{2.27}$$

Thus, together with $\tilde{u}^\mu$, they form a complete set transverse to q^μ, and therefore it is possible to express $\tilde{u}'^\mu$ in terms of them. The desired relation, which follows from substituting Eq. (2.8) into Eq. (2.24), is

$$\tilde{u}'_\mu = \sqrt{N_1} e_{1\mu} + \left(\frac{\tilde{u} \cdot u'}{\tilde{u}^2}\right) \tilde{u}_\mu , \tag{2.28}$$

which substituting in the definitions given in Eq. (2.8) yields the convenient formulas

$$\begin{aligned}
Q_{\mu\nu}(q, u') &= \frac{N_1}{\tilde{u}'^2} e_{1\mu} e_{1\nu} + \left(\frac{N_1}{\tilde{u}'^2} + 1\right) Q_{\mu\nu}(u, q) + \sqrt{N_1} \frac{\tilde{u} \cdot u'}{\tilde{u}^2 \tilde{u}'^2} (e_{1\mu} \tilde{u}_\nu + \tilde{u}_\mu e_{1\nu}) \\
P_{\mu\nu}(q, u') &= \frac{\mathcal{Q} \tilde{u} \cdot u'}{\mathcal{Q}' \tilde{u}^2} P_{\mu\nu}(q, u) + \frac{i\mathcal{Q}\sqrt{N_1}}{\mathcal{Q}' \tilde{u}^2} \left(\tilde{u}_\mu e_{2\nu} - \tilde{u}_\nu e_{2\mu}\right) .
\end{aligned} \tag{2.29}$$

On the other hand, as shown in Ref. [11], $R_{\mu\nu}$ can be decomposed in the form

$$R_{\mu\nu}(q, u) = -(e^\mu_1 e^\nu_1 + e^\mu_2 e^\nu_2) . \tag{2.30}$$

In this way, using Eqs. (2.23) and (2.29) in Eq. (2.22) together with the decomposition given in Eq. (2.30), we finally arrive at

$$\begin{aligned}
\pi^{(eff)}_{\mu\nu} = &-e_{1\mu} e_{1\nu} \left[\pi^{(m)}_T + \pi^{(f')}_T - \frac{N_1}{\tilde{u}'^2} \left(\pi^{(f')}_L - \pi^{(f')}_T\right)\right] - e_{2\mu} e_{2\nu} \left(\pi^{(m)}_T + \pi^{(f')}_T\right) \\
&+ \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} \left[\pi^{(m)}_L + \pi^{(f')}_L + \frac{N_1}{\tilde{u}'^2} \left(\pi^{(f')}_L - \pi^{(f')}_T\right)\right] + \sqrt{N_1} \frac{(\tilde{u} \cdot u')}{\tilde{u}^2 \tilde{u}'^2} \left(\pi^{(f')}_L - \pi^{(f')}_T\right) (e_{1\mu} \tilde{u}_\nu + \tilde{u}_\mu e_{1\nu}) .
\end{aligned} \tag{2.31}$$

Eq. (2.31), besides unfolding the main structure of the modes in a particularly simple way, is a useful formula that allows us to obtain the dispersion relation of the modes under a variety of conditions. In the absence of the stream, the solutions consist of one longitudinal mode with polarization vector $e^\mu_3 \propto \tilde{u}^\mu$, and two degenerate transverse modes with polarization vectors $e^\mu_{1,2}$ that satisfy $Q_{\mu\nu} e^\nu_{1,2} = 0$. Their dispersion relations are determined by solving the equations $q^2 = \mathrm{Re}\, \pi^{(m)}_{L,T}$ for the longitudinal and transverse modes, respectively. The presence of the stream breaks the degeneracy of the transverse modes, and in general causes a mixing between them with the longitudinal mode. In those cases in which it is permissible to treat the mixing term (the last term in Eq. (2.31)) as a perturbation (e.g., the number density in the stream is sufficiently smaller than those in the matter background), the dispersion relations are obtained approximately by solving

$$\begin{aligned}
q^2 &= \pi^{(m)}_T + \pi^{(f')}_T + \frac{N_1 q^2}{\mathcal{Q}'^2} \left(\pi^{(f')}_L - \pi^{(f')}_T\right) \\
q^2 &= \pi^{(m)}_T + \pi^{(f')}_T \\
q^2 &= \pi^{(m)}_L + \pi^{(f')}_L - \frac{N_1 q^2}{\mathcal{Q}'^2} \left(\pi^{(f')}_L - \pi^{(f')}_T\right) .
\end{aligned} \tag{2.32}$$

with corresponding polarization vectors $e_{1,2}$ and $e_3 \propto \tilde{u}$, respectively. In Eq. (2.32) we have used the relation $\tilde{u}^2 = -\mathcal{Q}^2/q^2$ and the corresponding one for $\tilde{u}'^2$. If the mixing term is not sufficiently small so that the higher order terms are important, then the full 2×2 problem in the $e_1 - \tilde{u}$ plane must be considered which, although tedious, is straightforward.

In the equations Eq. (2.32), it is understood that the variables $\Omega', \mathcal{Q}'$ are to be expressed in terms of $\Omega, \mathcal{Q}$ by means of the relations given in Eq. (2.21). They thus become implicit equations for $\Omega, \mathcal{Q}$, whose solutions determine the dispersion relations $\Omega(\mathcal{Q})$ of the various modes.

C. Discussion

For illustrative purposes, we consider the specific case of a stream of electrons and a matter background that consists of an electron gas and a non-relativistic proton gas. We borrow from Ref. [10][see Eqs. (A5) and (A9)] the following results

$$
\begin{aligned}
B_f(\Omega, \mathcal{Q}) &= -\frac{1}{2}\int \frac{d^3\mathcal{P}}{(2\pi)^3}\frac{\vec{\mathcal{Q}}\cdot\nabla_{\mathcal{P}}(f_f(\mathcal{E}) + f_{\overline{f}}(\mathcal{E}))}{\Omega - \vec{v}_{\mathcal{P}}\cdot\vec{\mathcal{Q}}}\,,\\
A_f(\Omega, \mathcal{Q}) &= B_f(\Omega, \mathcal{Q}) + \frac{\Omega}{2}\int \frac{d^3\mathcal{P}}{(2\pi)^3}\frac{\vec{v}_{\mathcal{P}}\cdot\nabla_{\mathcal{P}}(f_f(\mathcal{E}) + f_{\overline{f}}(\mathcal{E}))}{\Omega - \vec{v}_{\mathcal{P}}\cdot\vec{\mathcal{Q}}}\,,
\end{aligned}
\tag{2.33}
$$

where $\mathcal{E} = \sqrt{\vec{\mathcal{P}}^2 + m_f^2}$, $\nabla_{\mathcal{P}}$ is the gradient operator with respect to the momentum $\vec{\mathcal{P}}$ and $\vec{v}_{\mathcal{P}} = \vec{\mathcal{P}}/\mathcal{E}$. As shown there, they are valid for values of q such that

$$
q/\langle\mathcal{E}\rangle \ll 1\,,
\tag{2.34}
$$

where $\langle\mathcal{E}\rangle$ stands for a typical average value of the energy of the fermions in the gas. For distribution functions that depend on $\vec{\mathcal{P}}$ only through $\mathcal{E}$, we can replace $\nabla_{\mathcal{P}} \to \vec{v}_{\mathcal{P}}\frac{\partial}{\partial\mathcal{E}}$ in Eq. (2.33). Several useful formulas follow from Eq. (2.33) in particular cases. For example, if the fermions are relativistic,

$$
\begin{aligned}
A_f(\Omega, \mathcal{Q}) &= -3\omega_{0f}^2\\
B_f(\Omega, \mathcal{Q}) &= -3\omega_{0f}^2\left(1 - \frac{\Omega}{2\mathcal{Q}}\ln\left|\frac{\Omega + \mathcal{Q}}{\Omega - \mathcal{Q}}\right|\right)\,.
\end{aligned}
\tag{2.35}
$$

Eq. (2.35) holds for a degenerate or non-degenerate gas. For the non-relativistic and non-degenerate case we use

$$
\begin{aligned}
A_f(\Omega, \mathcal{Q}) &= -3\omega_{0f}^2 + \frac{\mathcal{Q}^2\omega_{0f}^2}{\Omega^2}\\
B_f(\Omega, \mathcal{Q}) &= \frac{\mathcal{Q}^2\omega_{0f}^2}{\Omega^2}\,,
\end{aligned}
\tag{2.36}
$$

which are valid if, in addition to Eq. (2.34),

$$
\Omega \gg \bar{v}_f\mathcal{Q}\,,
\tag{2.37}
$$

where $\bar{v}_f \equiv 1/\sqrt{\beta_f m_f}$ is a typical value of the velocity of the fermions in the gas. The quantity ω_{0f}^2, which is related to the plasma frequency in the gas, is given by

$$
\omega_{0f}^2 = \int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}}(f_f(\mathcal{E}) + f_{\bar{f}}(\mathcal{E}))\left[1 - \frac{\mathcal{P}^2}{3\mathcal{E}^2}\right]\,.
\tag{2.38}
$$

In Eqs. (2.35) and (2.36) we have used its form in the relativistic (ER) limit and non-relativistic (NR) limits,

$$
\omega_{0f}^2 = \begin{cases} \frac{1}{6\pi^2}\int_0^\infty d\mathcal{P}\mathcal{P}(f_f(\mathcal{P}) + f_{\bar{f}}(\mathcal{P})) & \text{(ER)}\\ \frac{n_f}{4m_f} & \text{(NR)}\,, \end{cases}
\tag{2.39}
$$

where n_f is the fermion number density in the frame in which the background is at rest. The corresponding formulas for the non-relativistic and degenerate case are given in Ref. [20].

Under most circumstances, the protons make a negligible contribution to the dispersion relations. The conditions under which those terms can be important are given in Ref. [20]. Here we do not include those special cases and therefore we will not consider the protons further. Lastly, we assume that the stream is not moving too fast as a whole, so that the term that is proportional to N_1 in Eq. (2.32) can be neglected, since according to Eq. (2.26) it is or the order of the velocity squared of the stream.

With these assumptions, the dispersion relations become

$$
\begin{aligned}
q^2 &= -2e^2\left[\left(A_e(\Omega, \mathcal{Q}) + \frac{q^2}{\mathcal{Q}^2}B_e(\Omega, \mathcal{Q})\right) + \left(A_{e'}(\Omega', \mathcal{Q}') + \frac{q^2}{\mathcal{Q}'^2}B_{e'}(\Omega', \mathcal{Q}')\right)\right]\\
q^2 &= 4e^2\left[\frac{q^2}{\mathcal{Q}^2}B_e(\Omega, \mathcal{Q}) + \frac{q^2}{\mathcal{Q}'^2}B_{e'}(\Omega', \mathcal{Q}')\right]\,,
\end{aligned}
\tag{2.40}
$$

for the transverse and longitudinal modes, respectively. We now consider several cases separately.

For the electrons in the matter background we use Eq. (2.36). Similarly, for the stream

$$A_{e'}(\Omega', \mathcal{Q}') = -3\omega_{0e'}^2 + \frac{\mathcal{Q}'^2\omega_{0e'}^2}{\Omega'^2}$$
$$B_{e'}(\Omega', \mathcal{Q}') = \frac{\mathcal{Q}'^2\omega_{0e'}^2}{\Omega'^2}\,, \tag{2.41}$$

which, as we will see, are suitable for finding the long wavelength limit of the dispersion relations. Eq. (2.41) can be used if the solution is such that

$$\Omega' \gg \bar{v}_{e'}\mathcal{Q}'\,. \tag{2.42}$$

The conditions under which the solution thus found is valid can be ascertained afterwards. The formula for $\omega_{0e'}^2$ is the same expression given in Eq. (2.38), but with replacement $f_{f,\bar{f}}(\mathcal{E}) \to f_{e',\bar{e}'}(\mathcal{E})$, where

$$f_{e',\bar{e}'}(\mathcal{E}) = \frac{1}{e^{\beta_{e'}\mathcal{E}\mp\alpha_{e'}} + 1}\,. \tag{2.43}$$

As we have indicated previously, the implicit assumption that is being made here is that the electrons that compose the stream have, in the rest frame of the stream, an isotropic thermal distribution characterized by a temperature and chemical $1/\beta_{e'}$ and $\alpha_{e'}$.

Let us consider the dispersion relation for the longitudinal mode. From Eqs. (2.15) and (2.18) this yields

$$1 = 4e^2\left(\frac{\omega_{0e}^2}{\Omega^2} + \frac{\omega_{0e'}^2}{\Omega'^2}\right)\,. \tag{2.44}$$

From Eq. (2.20)

$$\Omega' = \Omega - \vec{\mathcal{Q}}\cdot\vec{U}' \tag{2.45}$$

using $U'^0 \simeq 1$, and therefore the dispersion relation is

$$(\Omega - \vec{\mathcal{Q}}\cdot\vec{U}')^2(\Omega^2 - 4e^2\omega_{0e}^2) - 4e^2\omega_{0e'}^2\Omega^2 = 0\,. \tag{2.46}$$

The salient feature of Eq. (2.46) is that, besides the usual solution $\Omega_L^2(\mathcal{Q}\to 0) \simeq 4e^2\omega_{0e}^2$, there is another one with $\Omega_L \simeq \vec{U}'\cdot\vec{\mathcal{Q}}$. The standard way to find this second solution is to substitute

$$\Omega = \vec{U}'\cdot\vec{\mathcal{Q}} + \delta_L \tag{2.47}$$

in Eq. (2.46) and determine δ_L approximately by taking it to be a small quantity. In this fashion, we find

$$\delta_L = \pm\frac{|2e\omega_{0e'}\vec{U}'\cdot\vec{\mathcal{Q}}|}{\sqrt{(\vec{U}'\cdot\vec{\mathcal{Q}})^2 - 4e^2\omega_{0e}^2}}\,, \tag{2.48}$$

which shows the well-known instability of this system. For values of $\vec{U}'\cdot\vec{\mathcal{Q}}$ such that

$$0 < |\vec{U}'\cdot\vec{\mathcal{Q}}| < 2|e|\omega_{0e}\,, \tag{2.49}$$

the dispersion relation has a solution with a positive imaginary part, which signals that the system is unstable against oscillations with those values of $\vec{U}'\cdot\vec{\mathcal{Q}}$. The condition that δ_L be small relative to $\vec{U}'\cdot\vec{\mathcal{Q}}$ is satisfied for sufficiently small values of $\omega_{0e'}/\omega_{0e}$. On the other hand notice that, for this solution, $\Omega' = \delta_L$, and $\mathcal{Q}' = \sqrt{\Omega'^2 - \Omega^2 + \mathcal{Q}^2} \simeq \mathcal{Q}$. The conditions given in Eqs. (2.34) and (2.42) are then equivalent to $|\vec{U}'\cdot\vec{\mathcal{Q}}| \gg \bar{v}_e\mathcal{Q}$ and $\delta \gg \bar{v}_{e'}\mathcal{Q}$ which, for sufficiently small values of the thermal velocities, are satisfied as well.

Turning now the attention to the transverse dispersion relation, we substitute the formulas for $A_{e,e'}$ and $B_{e,e'}$ given in Eqs. (2.36) and (2.41) into Eq. (2.40). This yields simply

$$q^2 = 4e^2\omega_{0e}^2 + 4e^2\omega_{0e'}^2\,, \tag{2.50}$$

which shows that in this case the presence of the stream perturbs somewhat the transverse dispersion relation by shifting the value of the plasma frequency, but it does not produce a significant effect otherwise.

Eqs. (2.44) and (2.50) reproduce the well-known results found in textbooks [21], which are derived by kinetic theory or similar semi-classical methods. However, the results that we have obtained, and which are summarized in Eqs. (2.31) and (2.32), go farther. Together with the expressions for the self-energy functions in terms of the coefficients A_f and B_f [Eqs. (2.15) and (2.18)] they allow us to study systems under a wider variety of conditions, including those for which the semi-classical approaches, and the simple formula given in Eq. (2.44) in particular, are not be applicable.

2. Non-relativistic matter electrons and relativistic stream electrons

For the electrons in the stream we must use in this case

$$\begin{aligned} A_{e'}(\Omega', \mathcal{Q}') &= -3\omega_{0e'}^2 \\ B_{e'}(\Omega', \mathcal{Q}') &= -3\omega_{0e'}^2 \left(1 - \frac{\Omega'}{2\mathcal{Q}'} \ln\left|\frac{\Omega' + \mathcal{Q}'}{\Omega' - \mathcal{Q}'}\right|\right) , \end{aligned} \tag{2.51}$$

while the matter electron formulas are the same as the previous ones. The dispersion relations are then determined by

$$\Omega^2 - 4e^2\omega_{0e}^2 = 4e^2\omega_{0e'}^2 f_L \tag{2.52}$$

$$\Omega^2 - \mathcal{Q}^2 - 4e^2\omega_{0e}^2 = 4e^2\omega_{0e'}^2 f_T \tag{2.53}$$

for the longitudinal and transverse modes, respectively, where we have defined

$$\begin{aligned} f_L &= \frac{3\Omega^2}{\mathcal{Q}'^2}\left[\frac{\Omega'}{2\mathcal{Q}'} \ln\left|\frac{\Omega' + \mathcal{Q}'}{\Omega' - \mathcal{Q}'}\right| - 1\right] \\ f_T &= \frac{3}{2}\left\{1 + \frac{q^2}{\mathcal{Q}'^2}\left[1 - \frac{\Omega'}{2\mathcal{Q}'} \ln\left|\frac{\Omega' + \mathcal{Q}'}{\Omega' - \mathcal{Q}'}\right|\right]\right\} . \end{aligned} \tag{2.54}$$

Let us consider the longitudinal dispersion relation.

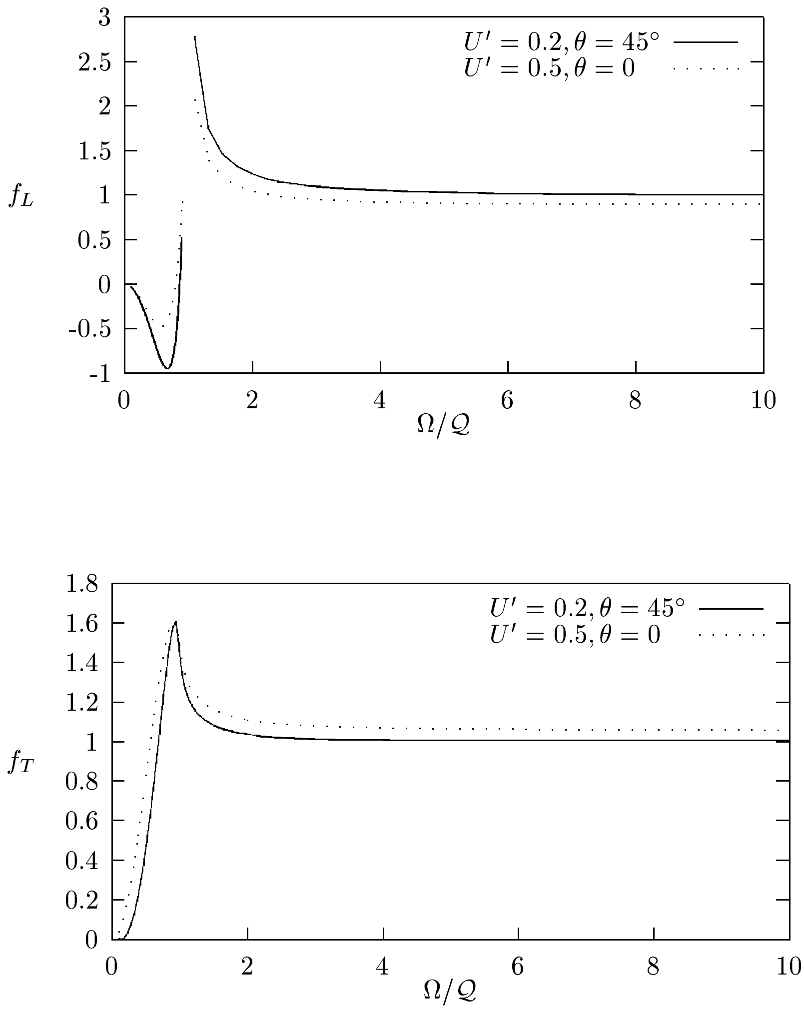

FIG. 2. The functions $f_{L,T}$ defined in the text, are plotted as functions of $\Omega/\mathcal{Q}$, for some representative values of the velocity of the stream U' and the angle between $\vec{\mathcal{Q}}$ and $\vec{U}'$. For $\Omega/\mathcal{Q} = 1$ the function f_L becomes infinite, while $f_T = 3/2$.

With the help of Fig. 2 it is easy to see that, besides the usual solution $\Omega_L \simeq 4e^2\omega_{0e}$, any other solution to Eq. (2.52) must have $\Omega \approx \mathcal{Q}$ (which implies that $\Omega' \approx \mathcal{Q}'$ and the function f_L can be large) so that the stream term can compete with the matter term in that equation. Substituting

$$\Omega = \mathcal{Q} + \delta_L \tag{2.55}$$

in Eq. (2.52) we find

$$\delta_L = \pm 2\mathcal{Q} e^{-2(\mathcal{Q}^2 - 4e^2\omega_{0e}^2)/12e^2\omega_{0e}^2}, \tag{2.56}$$

for $\mathcal{Q}^2 > 4e^2\omega_{0e}^2$. In contrast to the case considered in Section II C 1, there is no sign that a stream instability may develop in the present one.

For the transverse dispersion relations the situation is different. The function f_T is not larger than a number of order 1, as shown in Fig. 2, so that the stream contribution in Eq. (2.53) only introduces a perturbation in the usual solution.

In summary, when the stream consists of a relativistic electron gas, there is no sign that any stream instabilities may develop. This result will be a useful reference point when we consider in Section III the case in which the stream consists of neutrinos.

III. PHOTON SELF-ENERGY IN A NEUTRINO STREAM

In this section we consider a system composed of a matter background as in Section II B, and a neutrino stream with a velocity four-vector u'^{μ}. Our immediate task is to determine the neutrino stream contribution to the photon self-energy, for which we must calculate a diagram similar to the one in Fig. 1, but with a neutrino as the fermion in the loop. Denoting the effective electromagnetic vertex of the neutrino in the matter background by $\Gamma^{(\nu)}_{\mu}(q)$, then

$$i\pi^{(\nu)}_{11\mu\nu} = (-1)(-i)^2 \mathrm{Tr} \int \frac{d^4p}{(2\pi)^4} \Gamma^{(\nu)}_{\mu}(q) i S^{(\nu)}_{F11}(p+q) \Gamma^{(\nu)}_{\nu}(-q) i S^{(\nu)}_{F11}(p)\,, \tag{3.1}$$

where the neutrino propagator $S^{(\nu)}_{F11}$ is is given by Eq. (2.3) (with $m_\nu = 0$). The neutrino effective vertex can be expressed in the form

$$\Gamma^{(\nu)}_{\mu}(q) = T_{\mu\nu}(q)\gamma^{\nu} L\,, \tag{3.2}$$

where $L = \frac{1}{2}(1-\gamma_5)$ as usual, and $T_{\mu\nu}$ can be decomposed as

$$T_{\mu\nu} = T_T R_{\mu\nu}(q,u) + T_L Q_{\mu\nu}(q,u) + T_P P_{\mu\nu}(q,u)\,. \tag{3.3}$$

A detailed calculation of the various terms in Eq. (3.3) was carried out in Ref. [10]. For our purposes here, we can summarize the main results obtained there as follows.

For practical purposes, the contribution to $T_{T,L}$ due to the anomalous magnetic moment couplings of the nucleons in the background is negligible, so that

$$\begin{aligned} T_T &= 2\sqrt{2}|e|G_F a_p \left(A_p(\Omega,\mathcal{Q}) - \frac{B_p(\Omega,\mathcal{Q})}{\tilde{u}^2} \right) + T^{(e)}_T\,, \\ T_L &= 4\sqrt{2}|e|G_F a_p \frac{B_p(\Omega,\mathcal{Q})}{\tilde{u}^2} + T^{(e)}_L\,, \end{aligned} \tag{3.4}$$

while

$$T_P = T^{(e)}_P - 4\sqrt{2} G_F b_p \mathcal{Q}(|e| + 2m_p\kappa_p) C_p(\Omega,\mathcal{Q}) - 8 m_n \kappa_n \sqrt{2} G_F b_n \mathcal{Q} C_n(\Omega,\mathcal{Q})\,. \tag{3.5}$$

In these formulas

$$\begin{aligned} \kappa_p &= 1.79 \left(\frac{|e|}{2m_p} \right)\,, \\ \kappa_p &= -1.91 \left(\frac{|e|}{2m_n} \right)\,, \end{aligned} \tag{3.6}$$

are the anomalous magnetic moment of the nucleons, the coefficients a_f and b_f are the neutral current couplings of the fermion f, while A_p and B_p are defined in Eq. (2.12) and

$$C_f(\Omega,\mathcal{Q}) = \int \frac{d^3p}{(2\pi)^3 2E} \left(\frac{\tilde{u}\cdot p}{\tilde{u}^2} \right) (f_f - f_{\bar{f}}) \left[\frac{1}{q^2 + 2p\cdot q} + (q \to -q) \right]\,. \tag{3.7}$$

The electron terms $T^{(e)}_X$ were calculated in Ref. [7], and are given by

$$\begin{aligned} T^{(e)}_T &= 2\sqrt{2} e G_F \left(A_e(\Omega,\mathcal{Q}) - \frac{B_e(\Omega,\mathcal{Q})}{\tilde{u}^2} \right) \begin{cases} a_e + 1 & (\nu_e) \\ a_e & (\nu_{\mu,\tau}) \end{cases} \\ T^{(e)}_L &= 4\sqrt{2} e G_F \frac{B_e(\Omega,\mathcal{Q})}{\tilde{u}^2} \begin{cases} a_e + 1 & (\nu_e) \\ a_e & (\nu_{\mu,\tau}) \end{cases} \\ T^{(e)}_P &= -4\sqrt{2} e G_F \mathcal{Q} C_e(\Omega,\mathcal{Q}) \begin{cases} b_e - 1 & (\nu_e) \\ b_e & (\nu_{\mu,\tau}) \end{cases}\,. \end{aligned} \tag{3.8}$$

Substituting Eq. (3.3) in Eq. (3.1), we then obtain for the neutrino stream contribution to the photon self-energy

$$\mathrm{Re}\,\pi^{(\nu)}_{\mu\nu} = -2T_{\mu\alpha}(q)T_{\nu\beta}(-q)J^{\alpha\beta} \tag{3.9}$$

where

$$\begin{aligned} J^{\alpha\beta} \equiv \int \frac{d^3p}{(2\pi)^3 2E} \Bigg\{ & (f_\nu(p\cdot u') + f_{\bar\nu}(p\cdot u')) \left[\frac{2p^\alpha p^\beta - p^\alpha q^\beta - q^\alpha p^\beta + g^{\alpha\beta} p\cdot q}{q^2 - 2p\cdot q} + (q \to -q)\right] \\ & + (f_\nu(p\cdot u') + f_{\bar\nu}(p\cdot u'))\, i\epsilon^{\alpha\beta\lambda\rho} q_\lambda p_\rho \left[\frac{1}{q^2 - 2p\cdot q} + \frac{1}{q^2 - 2p\cdot q}\right] \Bigg\} \,. \end{aligned} \tag{3.10}$$

The transversality and symmetry properties of $J^{\alpha\beta}$ imply that it is expressible in terms of the tensors $R^{\alpha\beta}(q,u')$, $Q^{\alpha\beta}(q,u')$ and $P^{\alpha\beta}(q,u')$, with coefficients that can be determined by projecting Eq. (3.10) along these tensors. Thus we find

$$\begin{aligned} J^{\alpha\beta} = & \frac{1}{2}\left(A_\nu(\Omega',\mathcal{Q}') - \frac{B_\nu(\Omega',\mathcal{Q}')}{\tilde u'^{\,2}}\right) R^{\alpha\beta}(q,u') \\ & + \frac{B_\nu(\Omega',\mathcal{Q}')}{\tilde u'^{\,2}} Q^{\alpha\beta}(q,u') + \mathcal{Q}' C_\nu(\Omega',\mathcal{Q}') P^{\alpha\beta}(q,u')\,, \end{aligned} \tag{3.11}$$

with the coefficients $A_\nu(\Omega',\mathcal{Q}')$, $B_\nu(\Omega',\mathcal{Q}')$ and $C_\nu(\Omega',\mathcal{Q}')$ defined in Eqs. (2.12) and (3.7). In Eq. (3.11) the tensors $R_{\mu\nu}(q,u')$, $Q_{\mu\nu}(q,u')$ and $P_{\mu\nu}(q,u')$ are eliminated using Eqs. (2.23) and (2.29), and $R_{\mu\nu}(q,u)$ is decomposed as in Eq. (2.30). In this way,

$$\begin{aligned} J_{\alpha\beta} = & -\left\{\frac{1}{2}\left(A_\nu - \frac{B_\nu}{\tilde u'^{\,2}}\right) - \frac{N_1}{2\tilde u'^{\,2}}\left[\frac{3B_\nu}{\tilde u'^{\,2}} - A_\nu\right]\right\} e_{1\alpha}e_{1\beta} - \frac{1}{2}\left(A_\nu - \frac{B_\nu}{\tilde u'^{\,2}}\right) e_{2\alpha}e_{2\beta} \\ & + \left\{\frac{B_\nu}{\tilde u'^{\,2}} + \frac{N_1}{2\tilde u'^{\,2}}\left[\frac{3B_\nu}{\tilde u'^{\,2}} - A_\nu\right]\right\} Q_{\alpha\beta}(q,u) + C_\nu \mathcal{Q}\left(\frac{\tilde u\cdot u'}{\tilde u'^{\,2}}\right) P_{\alpha\beta}(q,u) \\ & + \frac{\sqrt{N_1}\,\tilde u\cdot u'}{2\tilde u'^{\,2}\tilde u^2}\left[\frac{3B_\nu}{\tilde u'^{\,2}} - A_\nu\right](e_{1\alpha}\tilde u_\beta + \tilde u_\alpha e_{1\beta}) + \frac{iC_\nu\sqrt{N_1}\mathcal{Q}}{\tilde u^2}(\tilde u_\alpha e_{2\beta} - e_{2\alpha}\tilde u_\beta)\,. \end{aligned} \tag{3.12}$$

By substituting Eq. (3.12) in Eq. (3.9) we finally obtain the formula for the neutrino contribution which, when it is added to $\pi^{(m)}_{\mu\nu}$ to obtain the total photon self-energy, is the starting point to determine the photon dispersion relations. However, with all its generality, the formula is not particularly useful and therefore we consider below some specific situations of potential interest.

A. Longitudinal dispersion relation

As already seen in Section II B, in general the effects of the stream break the degeneracy of the transverse modes and also mixes them with the longitudinal one. When the latter effect is not too large, the longitudinal dispersion relation is obtained approximately by solving the equation

$$q^2 = \pi^{(m)}_L + \pi^{(\nu)}_l\,, \tag{3.13}$$

where

$$\pi^{(\nu)}_l \equiv \frac{\tilde u^\mu \tilde u^\nu}{\tilde u^2}\pi^{(\nu)}_{\mu\nu}\,. \tag{3.14}$$

Using the relation $\tilde u^\mu T_{\mu\alpha} = T_L \tilde u_\alpha$, together with

$$\begin{aligned} \frac{\tilde u^\mu \tilde u^\nu}{\tilde u^2} R_{\mu\nu}(q,u') &= -\frac{N_1}{\tilde u'^{\,2}} \\ \frac{\tilde u^\mu \tilde u^\nu}{\tilde u^2} Q_{\mu\nu}(q,u') &= \frac{N_1}{\tilde u'^{\,2}} + 1\,, \end{aligned} \tag{3.15}$$

we obtain from Eq. (3.9)

$$\mathrm{Re}\,\pi_l^{(\nu)} = 2T_L^2 \frac{q^2}{\mathcal{Q}'^2}\left\{B_\nu - N_1\left[\frac{3q^2B_\nu}{\mathcal{Q}'^2} + \frac{1}{2}A_\nu\right]\right\}\,, \tag{3.16}$$

where, to simplify the notation, we have omitted the arguments Ω' and $\mathcal{Q}'$ in the coefficients A_ν and B_ν. The dispersion relation obtained by substituting Eqs. (2.15) and (3.16) in Eq. (3.13) is the same as the corresponding one for a stream of charged particles, with an effective charge $e_\nu = \frac{1}{\sqrt{2}}T_L$.

1. Long wavelength limit

We consider the case analogous to the one discussed in Section II C, namely, a matter background made of a non-relativistic electron gas and a non-relativistic nucleon gas, and the neutrino stream. As in the case mentioned, the photon momentum is assumed to be such that Eq. (2.34) holds. For the electrons in the matter background we then use the formulas for $A_e(\Omega,\mathcal{Q})$ and $B_e(\Omega,\mathcal{Q})$ given in Eq. (2.36), which are valid for $\Omega \gg \bar{v}_e\mathcal{Q}$ as indicated in Eq. (2.37), while the analogous proton terms are negligible. From Eqs. (3.4) and (3.8), this yields

$$T_L = -\frac{q^2}{\Omega^2}T_0\,, \tag{3.17}$$

where

$$T_0 \equiv 4\sqrt{2}G_F e\omega_{0e}^2 \begin{cases} a_e + 1 & (\nu_e) \\ a_e & (\nu_{\mu,\tau}) \end{cases}\,. \tag{3.18}$$

The neutrinos are, for all practical purposes, ultrarelativistic. For them, we use the formulas analogous to those in Eq. (2.35), namely

$$\begin{aligned} A_\nu(\Omega',\mathcal{Q}') &= -3\omega_{0\nu}^2 \\ B_\nu(\Omega',\mathcal{Q}') &= -3\omega_{0\nu}^2\left(1 - \frac{\Omega'}{2\mathcal{Q}'}\ln\left|\frac{\Omega'+\mathcal{Q}'}{\Omega'-\mathcal{Q}'}\right|\right)\,, \end{aligned} \tag{3.19}$$

with

$$\omega_{0\nu}^2 = \frac{1}{6\pi^2}\int_0^\infty d\mathcal{P}\mathcal{P}(f_\nu(\mathcal{P}) + f_{\bar{\nu}}(\mathcal{P}))\,, \tag{3.20}$$

where

$$f_{\nu,\bar{\nu}}(\mathcal{P}) = \frac{1}{e^{\beta_\nu\mathcal{P}-\alpha_\nu}+1} \tag{3.21}$$

are the neutrino and antineutrino momentum distributions, in the rest frame of the stream. The longitudinal dispersion relation obtained from Eq. (3.13) is

$$\Omega^2 = 4e^2\omega_{0e}^2 + 2T_0^2\frac{(q^2)^2}{\Omega^2\mathcal{Q}'^2}\left\{B_\nu - N_1\left(\frac{3q^2}{\mathcal{Q}'^2}B_\nu + \frac{1}{2}A_\nu\right)\right\}\,. \tag{3.22}$$

In this equation, $\mathcal{Q}'$ and Ω' are expressed in terms of $\mathcal{Q}$ and Ω by means of $\mathcal{Q}' = \sqrt{\Omega'^2 - q^2}$ with $\Omega' = U'^0\Omega - \vec{U}'\cdot\vec{\mathcal{Q}}$, as indicated by Eq. (2.20). The solutions of Eq. (3.22) determine the dispersion relation $\Omega_L(\mathcal{Q})$ in the long wavelength limit and are valid for $\Omega \gg \bar{v}_e\mathcal{Q}$.

B. Neutrino driven stream instabilities

Besides the usual solution $\Omega_L^2 \simeq 4e^2\omega_{0e}^2$, Eq. (3.22) can have an additional solution if the neutrino term is sufficiently large that it can compete with the electron term. To determine whether this can happen, consider the specific situation in which the velocity of the neutrino stream is not too large, so that the term in Eq. (3.22) proportional to N_1 can be neglected [see Eq. (2.26)]. Using the formula for B_ν given in Eq. (3.19), the longitudinal dispersion relation then becomes

$$\Omega^2 - 4e^2\omega_{0e}^2 = T_0^2\omega_{0\nu}^2 \left(1 - \frac{\mathcal{Q}^2}{\Omega^2}\right)^2 f_L\,, \tag{3.23}$$

where f_L is the same function defined in Eq. (2.54). For values of $\Omega \approx \mathcal{Q}$ the function f_L becomes large, but with the factor $(1 - \mathcal{Q}^2/\Omega^2)$ its contribution in Eq. (3.23) is negligible in that region. On the other hand, for $\Omega \to 0$,

$$B_e(\Omega \to 0, \mathcal{Q}) = -\beta_e m_e \omega_{0e}^2 \tag{3.24}$$

so that

$$T_L = -T_0\beta_e m_e \tag{3.25}$$

in this limit, instead of Eq. (3.17). Therefore, the neutrino contribution is not large in the limit $\Omega \to 0$ either.

We therefore conclude that the neutrino contribution produces a small correction to the usual dispersion relation but does not introduce any additional branch. In particular, there are no stream-induced instabilities in this system. This conclusion is in sharp contradiction with the result obtained in Ref. [15], where it was found that, in the same system, the dispersion relation indicates the appearance of neutrino-driven stream instabilities.

To understand the origin of this discrepancy it is useful to consider

$$f_{\nu,\bar{\nu}} = (2\pi)^2 n_{\nu,\bar{\nu}}\delta^{(3)}(\vec{\mathcal{P}} - \mathcal{E}\hat{U}') \tag{3.26}$$

for the momentum distribution function of the neutrinos, which is of the form used in Ref. [15]. Using it in Eq. (2.33) to calculate B_ν results in

$$B_\nu = \left(\frac{n_\nu + n_{\bar{\nu}}}{2\mathcal{E}}\right) \frac{\mathcal{Q}^2 - (\vec{\mathcal{Q}}\cdot\hat{U}')^2}{(\Omega - \vec{\mathcal{Q}}\cdot\hat{U}')^2}\,, \tag{3.27}$$

which, when substituted in Eq. (3.22), yields the longitudinal dispersion relation

$$\Omega^2(\Omega^2 - 4e^2\omega_{0e}^2) = \frac{T_0^2(n_\nu + n_{\bar{\nu}})}{\mathcal{E}} \left(\frac{\Omega^2 - \mathcal{Q}^2}{\mathcal{Q}'}\right)^2 \frac{\mathcal{Q}^2\sin^2\theta}{(\Omega - \mathcal{Q}\cos\theta)^2}\,, \tag{3.28}$$

where $\cos\theta = \vec{\mathcal{Q}}\cdot\hat{U}'$ and we have neglected the term proportional to N_1, as before. Eq. (3.28) has a solution of the form

$$\Omega_L = \mathcal{Q}\cos\theta + \delta_L^{(\nu)}\,, \tag{3.29}$$

with

$$\delta_L^{(\nu)\,2} = \frac{T_0^2(n_\nu + n_{\bar{\nu}})}{\mathcal{E}} \frac{\mathcal{Q}^2\sin^4\theta}{\cos^2\theta(\mathcal{Q}^2\cos^2\theta - 4e^2\omega_{0e}^2)}\,, \tag{3.30}$$

which exhibits an instability for $\mathcal{Q}^2\cos^2\theta < 4e^2\omega_{0e}^2$.

Thus, while we are able to reproduce qualitatively the result of Ref. [15] in this way, we must note that it is based on an inconsistent application of the long wavelength formulas given in Eq. (2.33). As explained in detail in Ref. [20], those formulas are obtained by expanding the integrands in powers of $q/\mathcal{E}$ in the one-loop formulas given in Eq. (2.12), and retaining only those terms that are dominant in the limit $q/\mathcal{E} \to 0$. This requires, among other conditions, that the momentum distribution functions be such that its derivatives do not introduce any singularities in the integrands. The results derived in this way are equivalent to those obtained by semiclasical methods based on kinetic theory or similar approaches. However, the form given in Eq. (3.26) does not satisfy the required conditions and therefore neither the long wavelength approximation of the one-loop formulas, nor the semiclassical formulas, are applicable in that case.

Leaving aside the question of whether or not a distribution function such as that given in Eq. (3.26) is realistic in any particular physical context, in order to use it the coefficients A_ν, B_ν must be calculated with the complete one-loop formulas given in Eq. (2.12). Following this procedure we obtain

$$B_\nu = 2\mathcal{E}(n_\nu + n_{\bar\nu})\left[\frac{\mathcal{Q}^2 - (\vec{\mathcal{Q}}\cdot\vec{U}')^2}{4\mathcal{E}^2(\Omega - \vec{\mathcal{Q}}\cdot\vec{U}')^2 - (\Omega^2 - \mathcal{Q}^2)^2}\right] \tag{3.31}$$

instead of Eq. (3.27), so that the longitudinal dispersion relation becomes

$$\Omega^2(\Omega^2 - 4e^2\omega_{0e}^2) = 4T_0^2\mathcal{E}(n_\nu + n_{\bar\nu})\left(\frac{\Omega^2 - \mathcal{Q}^2}{\mathcal{Q}'}\right)^2 \frac{\mathcal{Q}^2\sin^2\theta}{4\mathcal{E}^2(\Omega - \mathcal{Q}\cos\theta)^2 - (\Omega^2 - \mathcal{Q}^2)^2}\,. \tag{3.32}$$

If we neglect here the q^2 term in the denominator, we recover Eq. (3.28). However, when that term is taken into account, the neutrino contribution does not become large for $\Omega \approx \vec{\mathcal{Q}}\cdot\vec{U}'$, and therefore a solution of the form given in Eq. (3.29) does not exist.

C. Transverse dispersion relation

The dispersion relations for the transverse modes are given approximately by solving the equation

$$\left[(q^2 - \pi_T^{(m)})R_{\mu\nu}(q,u) - \pi^{(\nu)}_{t\mu\nu}\right] A^\nu = 0\,, \tag{3.33}$$

where

$$\begin{aligned}\pi^{(\nu)}_{t\mu\nu} &\equiv R_\mu{}^\alpha(q,u)R_\nu{}^\beta(q,u)\pi^{(\nu)}_{\alpha\beta}\\ &= -2\left[T_T(q)R_{\mu\lambda}(q,u) + T_P(q)P_{\mu\lambda}(q,u)\right]\left[T_T(q)R_{\nu\rho}(q,u) - T_P(q)P_{\nu\rho}(q,u)\right]J^{\lambda\rho}\end{aligned} \tag{3.34}$$

is the transverse projection of the neutrino contribution to the self-energy, and in the second line we have used Eq. (3.9). For $J^{\alpha\beta}$ we will use the formula given in Eq. (3.11) and consider the case in which the terms with the factor N_1 can be neglected, as we did in Section III B. Therefore, remembering Eq. (2.30), we will substitute in Eq. (3.34)

$$J^{\alpha\beta} = \frac{1}{2}\left(A_\nu - \frac{B_\nu}{\tilde{u}'^2}\right)R_{\alpha\beta}(q,u) + C_\nu\mathcal{Q}\left(\frac{\tilde{u}\cdot u'}{\tilde{u}^2}\right)P_{\alpha\beta}(q,u) \tag{3.35}$$

It is now useful to introduce the linear combinations [22]

$$R^{(\pm)}_{\alpha\beta} \equiv \frac{1}{2}\left(R_{\alpha\beta}(q,u) \pm P_{\alpha\beta}(q,u)\right)\,, \tag{3.36}$$

which satisfy

$$R^{(s)\alpha\beta}R^{(s')}_{\beta\gamma} = \delta_{ss'}\delta^\alpha_\gamma\,, \tag{3.37}$$

and have the representation

$$R^{(\pm)}_{\alpha\beta} = -e^{(\pm)}_\alpha e^{(\pm)\dagger}_\beta \tag{3.38}$$

where

$$e^{(\pm)}_\alpha = \frac{1}{\sqrt{2}}(e_{1\alpha} \pm ie_{2\alpha})\,. \tag{3.39}$$

Writing $R_{\alpha\beta}$ and $P_{\alpha\beta}$ in terms of $R^{(\pm)}_{\alpha\beta}$ and substituting the resulting formulas in Eq. (3.34), with the help of Eq. (3.37) we obtain

$$\pi^{(\nu)}_{t\mu\nu} = \pi^{(+)}R^{(+)}_{\mu\nu} + \pi^{(-)}R^{(-)}_{\mu\nu} \tag{3.40}$$

where

$$\pi^{(\pm)} = -2\left(T_T \pm T_P\right)^2\left[\frac{1}{2}\left(A_\nu - \frac{B_\nu}{\tilde{u}'^2}\right) \pm C_\nu\mathcal{Q}\left(\frac{\tilde{u}\cdot u'}{\tilde{u}^2}\right)\right]\,. \tag{3.41}$$

From Eq. (3.33), the dispersion relations for the transverse modes are then

$$q^2 = \pi_T^{(m)} + \pi^{(\pm)} \tag{3.42}$$

with the corresponding polarization vectors $e^{(\pm)}_\alpha$.

D. Optical Activity of the neutrino gas

Eq. (3.41) exhibits the phenomenon of optical activity of the neutrino gas, in which the two circularly polarized photon modes travel with different dispersion relations as a result of the chiral interactions of the neutrino [11]. Notice that for this occur, T_P and/or C_ν must be non-zero. This requires that the chemical potentials in the matter background be such that, for some particle species, $\alpha_f \neq 0$, or that $\alpha_\nu \neq 0$. In the latter case however, there is an additional contribution to the photon self-energy that arises from the set of diagrams that were calculated in detail in Ref. [12]. Those diagrams are not included in Fig. 1 and their result is an additional contribution to the photon self-energy that must be taken into account in Eq. (3.42). The result of the calculation of Ref. [12] is taken into account by including in the right-hand side of Eq. (3.9) the term

$$\Pi^{(\nu)}_P P_{\mu\nu}(q, u') \tag{3.43}$$

where, in the notation of the present paper,

$$\Pi^{(\nu)}_P = \frac{\sqrt{2} G_F \alpha}{3\pi} \frac{q^2}{m_e^2} (n_\nu - n_{\bar{\nu}}) \mathcal{Q}' \,, \tag{3.44}$$

with

$$n_{\nu,\bar{\nu}} = \int \frac{d^3\mathcal{P}}{(2\pi)^3} f_{\nu,\bar{\nu}}(\mathcal{P}) \,. \tag{3.45}$$

The result quoted in Eq. (3.44) is valid for values of $q < m_e$. When this term is included in Eq. (3.34), the net effect is that the right-hand side of Eq. (3.42) has the additional the term

$$\pi_P = \pm \Pi^{(\nu)}_P \left(\frac{\mathcal{Q}}{\mathcal{Q}'}\right) \frac{\tilde{u} \cdot u'}{\tilde{u}^2} \,. \tag{3.46}$$

If the background contains an equal number of neutrinos and antineutrinos, then $\Pi^{(\nu)}_P$ as well as C_ν are zero. In such a case, the optical activity of the neutrino gas is due to a non-zero value of T_P in Eq. (3.41), which in turn depends on the difference between the particle and antiparticle number densities in the matter background.

1. Long wavelength limit

$\pi^{(\pm)}$ can be evaluated explicitly by considering specific situations. As an example we consider once more a matter background that consists of non-relativistic electron proton gases, with the photon momentum satisfying Eq. (2.34) and $\Omega \gg \bar{v}_f \mathcal{Q}$. The proton contribution to T_T is negligible, and using Eq. (2.33) for A_e and B_e,

$$T_T = -T_0 \,, \tag{3.47}$$

with T_0 given in Eq. (3.18). On the other hand, T_P is given by Eq. (3.5) where, in the momentum regime that we are considering,

$$C_f(\Omega, \mathcal{Q} \to 0) = -\frac{1}{2} \int \frac{d^3\mathcal{P}}{(2\pi)^3 2\mathcal{E}} \left(\frac{f_f - f_{\bar{f}}}{\mathcal{E}}\right) \left[1 - \frac{2\mathcal{P}^2}{3\mathcal{E}^2}\right] , \tag{3.48}$$

as shown in Ref. [10]. For the electron and proton gases we use the non-relativistic limit of this, $C_f = -\omega^2_{0f}/2m_f$, which implies that the proton term is negligible and therefore

$$T_P = \left(\frac{\mathcal{Q}}{2m_e}\right) T'_0 \,, \tag{3.49}$$

where

$$T'_0 = 4\sqrt{2} G_F e \omega^2_{0e} \begin{cases} b_e - 1 & (\nu_e) \\ b_e & (\nu_{\mu,\tau}) \end{cases} \tag{3.50}$$

For the neutrino gas, the relativistic limit of Eq. (3.48) yields

$$C_\nu = -\frac{1}{24\pi^2}\int_0^\infty d\mathcal{P}(f_\nu(\mathcal{P}) - f_{\bar{\nu}}(\mathcal{P}))\,, \tag{3.51}$$

while A_ν and B_ν are given in Eq. (3.19).

With the help of these formulas and remembering that $\mathrm{Re}\,\pi_T^{(m)} = 4e^2\omega_{0e}^2$ in the case we are considering, the dispersion relation becomes

$$q^2 = 4e^2\omega_{0e}^2 + 2T_0^2\omega_{0\nu}^2\left[1 \mp \frac{\mathcal{Q}T_0'}{2m_eT_0}\right]^2\left[1 \mp \frac{C_\nu\mathcal{Q}}{\omega_{0\nu}^2}\left(\frac{\tilde{u}\cdot u'}{\tilde{u}^2}\right)\right]^2 \pm \frac{\Pi_P^{(\nu)}\mathcal{Q}}{\mathcal{Q}'}\left(\frac{\tilde{u}\cdot u'}{\tilde{u}^2}\right)\,, \tag{3.52}$$

where we have included the $\Pi_P^{(\nu)}$ term as indicated in Eq. (3.46).

Let us consider first the situation in which $f_\nu \approx f_{\bar{\nu}}$, so that $\Pi_P^{(\nu)}$ and C_ν can be neglected in Eq. (3.52). The solutions for the two modes are then given by

$$\Omega_\pm = \sqrt{\mathcal{Q}^2 + 4e^2\omega_{0e}^2} \mp \frac{T_0T_0'\omega_{0\nu}^2}{m_e}\frac{\mathcal{Q}}{\sqrt{\mathcal{Q}^2 + 4e^2\omega_{0e}^2}}\,, \tag{3.53}$$

which is of the same form as that given in Eq. (4.22) of Ref. [12], if we make the correspondence

$$\frac{1}{2}aR_\nu \rightarrow \frac{T_0T_0'\omega_{0\nu}^2}{m_e} \tag{3.54}$$

there. For the situations of potential interest analyzed in Ref. [12], the effects of the dispersion relations given in Eq. (3.53) are smaller than those found in that reference by a factor of about $G_F\omega_{0e}^2 \approx G_Fn_e/m_e$, and whence are unimportant for all practical purposes.

Therefore, retaining only the term proportional to $\Pi_P^{(\nu)}$ in Eq. (3.52), the dispersion relation becomes

$$q^2 = 4e^2\omega_{0e}^2 \pm \xi\Pi_P^{(\nu)}\,, \tag{3.55}$$

where

$$\xi = \frac{\mathcal{Q}}{\mathcal{Q}'}\left(\frac{\tilde{u}\cdot u'}{\tilde{u}^2}\right)\,. \tag{3.56}$$

Of course, when the neutrino gas is not moving relative to the matter background ($\vec{U}' = 0$), $\xi = 1$ and Eq. (3.55) reduces to the form given in Ref. [12].

The salient feature here is that, in general, the dispersion relation is not isotropic, so that the splitting between the two circularly polarized modes is different depending on the direction of propagation of the photon relative to the velocity of the neutrino gas. To assess the consequences that this effect can have on the analysis given in Ref. [12] we consider two extreme cases.

a. $\vec{\mathcal{Q}}$ perpendicular to $\vec{U}'$. Using $\vec{\mathcal{Q}}\cdot\vec{U}' = 0$, it is very simple to verify that

$$\tilde{u}\cdot u' = -\frac{U'^0\mathcal{Q}^2}{q^2}\,,$$

while $\mathcal{Q}' = \sqrt{\vec{U}'^2 + \mathcal{Q}^2}$. Using $\tilde{u}^2 = -\mathcal{Q}^2/q^2$, it then follows that

$$\xi = \frac{\mathcal{Q}\sqrt{1+\vec{U}'^2}}{\sqrt{\Omega^2\vec{U}'^2 + \mathcal{Q}^2}}\,. \tag{3.57}$$

For small velocities of the neutrino gas this reduces to 1, as it should be, while for large velocities it implies that the effect of the $\Pi_P^{(\nu)}$ term is reduced by the factor $\mathcal{Q}/\Omega$ for $\Omega > \mathcal{Q}$.

b. $\vec{\mathcal{Q}}$ *parallel to* $\vec{U}'$. We set

$$\vec{Q} = \lambda \mathcal{Q} \hat{U}' \tag{3.58}$$

to include the possibility that the photon propagates antiparallel to the velocity of the neutrino gas. A little algebra shows that in this case

$$\tilde{u} \cdot u' = \frac{-U'^0 \mathcal{Q}^2 + \lambda \Omega \mathcal{Q} |\vec{U}'|}{q^2}$$

while $\mathcal{Q}' = \left| \Omega |\vec{U}'| - \lambda \mathcal{Q} U'^0 \right|$. Therefore

$$\xi = \frac{\mathcal{Q} - \lambda \beta' \Omega}{|\Omega \beta' - \lambda \mathcal{Q}|}\,, \tag{3.59}$$

where we have defined the speed of the neutrino gas

$$\vec{\beta}' = \frac{\vec{U}'}{U'^0}\,. \tag{3.60}$$

Eq. (3.59) reveals in a simple way the anisotropic nature of the dispersion relation. For example, if the velocity of the neutrino stream is such that

$$\frac{\mathcal{Q}}{\Omega} < \beta'\,, \tag{3.61}$$

then $\xi = -1$ or $+1$ depending on whether the photon is propagating parallel or antiparallel to $\vec{\beta}'$, respectively. In particular, this implies that the frequency difference between the dispersion relations of the two (circularly polarized) transverse modes is the opposite to what it is if the neutrino gas is not moving relative to the matter background. This effect is easy to understand by noticing that, if the condition in Eq. (3.61) holds, then a photon moving parallel to $\vec{\beta}'$ appears to be moving in the opposite direction in the rest frame of the neutrino gas.

IV. OUTLOOK

Although our work has been largely motivated by the study of the electromagnetic properties of a neutrino gas that moves, as a whole, relative to a plasma, the approach is applicable to a wider class of problem in similar physical systems, involving relativistic plasmas or high temperature gauge theories. The field theory methods employed here allow us to consider situations for which the semiclassical approaches, such as those based on kinetic theory, are not suitable, and those for which the full power of the techniques and methods that have been developed for high temperature field theory calculations must be employed. Some possible extensions of the present work along those lines would involve the calculation of the imaginary part of the self-energy to determine the damping rates, and the application of the resummation methods [23] to study the dispersion relations in those circumstances in which the perturbative approximations are not reliable.

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$$i\pi^{(ab)}_{\mu\nu}(q) =$$